Daniel Muzzulini: The natural waveform of tones – some contributions to the story of acoustics by British sources (2006)

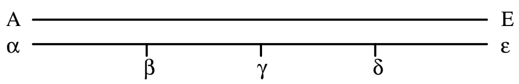
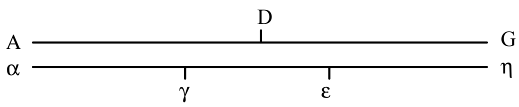
Daniel Muzzulini, Alemannengasse 35, CH-4058 Basel, ++41 61 691 59 16

Introduction

The preeminent role of the sine functions, the harmonic oscillations, usually remains unquestioned in the current discussions about the nature of tones.
It is already in the middle of the 18th century that Daniel Bernoulli asserts against Leonhard Euler that all periodic functions can be represented as superposed sine functions and that the overtones audible in many musical instruments are the sounding proofs of his theory. Georg Simon Ohm was the first to apply Fourier's Theorem [Ohm 1843; 1844] to acoustics by identifying the overtones with the sinusoidal components of complex periodic tones. And Hermann Helmholtz's resonance model of the basilar membrane [Helmholtz 1863], which establishes a correspondence between perceived simple tones and harmonic vibrations seems to leave no room for further questions.
Since the phenomenon of overtones was already known in the 17th century, a straightforward interpretation of the history of acoustics could argue that the overtones had to wait about 200 years to come to a convincing explanation of their existence. The present paper compares some ideas concerning the nature of tones found in British theoretical writings, namely by John Wallis [1676/67], Robert Smith [1749], Thomas Young [1800], and William Thomson 1877/78] with the contemporary mainstream acoustics.

1. John Wallis

In his *Letter to the editor* of the Transactions of the Royal Society from 1676/77 John Wallis reports the observation of *nodes*, i. e. places remaining at rest on a vibrating string set into resonance by a consonant sound of a different pitch. He also says that the phenomenon is already well known at that time in England.
After the frequency ratio of the octave (2 : 1) and the twelfth (3 : 1) he discusses the double octave (4 : 1) and, most interestingly, the fifth (3 : 2) in just intonation as follows:

> In like manner, if AE be a double Octave
>
> 
>
>
> to αε; the for quarters of *this* will tremble, when *that* is struck, but not the points, β, γ, δ. So if AG be a Fifth to αη; and
>
> 
>
>
> consequently each half of *that* stopped in D, an Unison to each third part of *this* stopped in γε; while *that* is struck, each part of *this* will tremble severally, but not the points γ, ε; and while *this* is struck, each of *that* will tremble; but not the point D. The like will hold in lesser concords; but the less remarkably as the number of divisions increases. […] This and the former I judge to depend upon one and the same cause; *viz*. the contemporary vibrations of the several Unison parts, which make the one tremble at the motion of the other: But when struck at the respective points of divisions, the sound is incongruous, by reason that the point is disturbed which should be at rest. [Wallis 1676/77, 841]

Let the pitches of the two sounds be *c* and *g*. From a modern viewpoint, which treats resonance as a phenomenon that occurs exclusively between two simple tones of equal pitch, Wallis' example can be explained by the coincidence of the second harmonic *g*' of *g* with the third harmonic of *c*, which is also *g*'. This argument seems to prove the complexity of the plucked string sounds as a by-product. Only if the exciting and the resonating string share a frequency component, can resonance take place and, since periodic signals consist of multiples of the fundamental frequency, must the string sound be compound.
Slightly differently and more carefully formulated, a freely vibrating string can have three or more different *natural frequencies* that are multiples of the fundamental frequency[1]. The nodes observed by Wallis prove the isolated overtones to the eye. Resonance is thereby reduced to a phenomenon between tone pairs of equal pitch, which Wallis justly recognizes in referring to "the contemporary vibrations of the several Unison parts". Wallis's description goes beyond the repeated proclamation of the complex nature of sound given by Marin Mersenne [1636] in this respect. The latter merely supports his assertion by referring to the attentive listener with a fine ear, whereas the former gives evidence by a congruent visual perception.
At the end of his letter Wallis discusses the resonance of a string induced by consonant sounds of an organ, which seems to prove the complexity of the generating organ sound in a similar way: the upper sound *g* of the fifth *c*–*g*, which is played by the organ, contains the harmonic *g*', which causes the string tuned to *c* to resonate at its third harmonic *g*'.
On this occasion, he points to the differences in the resonance behavior among strings of different material, an observation that supports a spectral theory of sound quality, in which differences in timbre are measured by differences in resonance behavior i.e. in terms of different sets of natural frequencies.
Wallis, however, does not develop such a spectral theory of sound quality. One reason for this might be that he is probably unfamiliar with the overtones. Leaving them apart, the nodes observed only demonstrate that different tones are possible in the same open string, but not necessarily that they can be excited simultaneously. Remarkably, Wallis does not report of the pitch of the resonating string sound, which can be clearly distinguished from its fundamental pitch, when the exciting string is damped after the onset of the resonance.

The given explanation assumes *stationary spectra*, whose sets of frequency components do non alter over time, in other words they are an inherent property of the sounding body. Such frequency components are called eigenfrequencies, member frequencies or natural frequencies. Only if their frequency sets have a non-empty intersection can mutual resonance between two sounding bodies occur. From this point of view, resonance is a strictly linear phenomenon, since there are no new frequencies generated. Although Mersenne recognizes the compound nature of many different sounds, the notion of stationary spectra as in the above argument is not yet developed in the 17th century. Without a distinction between simple and complex sounds, the frequency of the responding sound is to be explained as generated in a non-linear way, namely as the least common multiple of the two fundamental frequencies. In the example of the fifth $c = 2$ / $g = 3$ the least common multiple is $g' = 6$, a new sound. Arguments of this kind were later on brought-up by Mairan [1737] and Rameau [1737]. Since Wallis had neither Newton's laws nor sine-functions, nor Fourier's theorem at his disposal, the explanation by coinciding frequency components is rather anachronistic, and it has to be emphasized that in the 17th century, there is no property of a harmonic (sinusoidal) oscillation – as opposed to the motion of the freely vibrating string – that would justify calling its sound a simple or elementary tone.

Galileo Galilei is familiar with resonance between strings whose fundamental sounds form a perfect octave or a fifth:

> [...] the wonderful phenomenon of the strings of the cittern [*cetera*] or of the spinet [*cimbalo*], namely, the fact that a vibrating string will set another string in motion and cause it to sound not only when the latter is in unison but even when it differs from the former by an octave or a fifth. [Galilei 1638, 141–142]

However he gives only a thorough explanation of the unison case by coinciding pulses leading to reinforcement [141]. A difficulty, which comes up with the generalization of this argument from unisons to other consonances, is the interpretation of the non-coinciding pulses. Among the pulses at time 0, 2, 3 and 4 of one cycle of 6 time units only the pulses at time 0 coincide, whereas the other three unsynchronized pulses at 2, 3 and 4 could annihilate the effect of the 0-pulse, see Fig. 1. Galilei explains the consonance of the fifth by this pulse-pattern and concludes his argument as follows:

> Thus the effect of the fifth is to produce a tickling of the ear drum such that its softness is modified with sprightliness, giving at the same time the impression of a gentle kiss and of a bite. [Galilei 1638, 149]

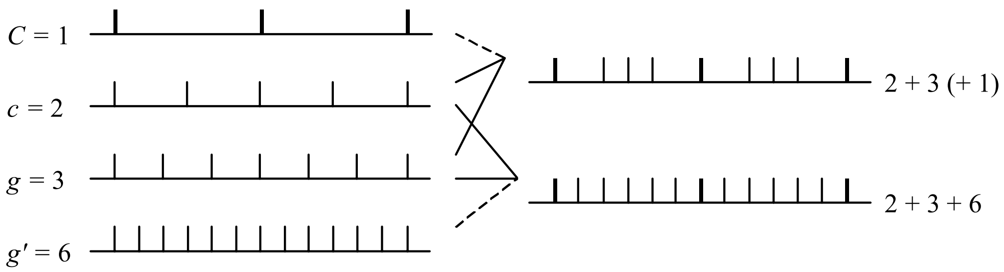


Fig. 1 The overlaid impulse patterns of the fifth *c* : *g* in just tuning result in a repetitive pattern of the same periodicity as the missing fundamental (*C* = 1). Adding the fundamental (in an appropriate phase) does not affect this pattern. Adding the resonating frequency *g′* = 6 however completes it to a regular pattern of a time period of one sixth of that of the fundamental.

Remarkably, Mersenne does not make explicit use of the concept of nodes even earlier. He is very close to awareness of it in his discussion of the *trumpet marine*, whose sound is produced by enforced nodes as flageolet tones, but he rejects his right explication of the simultaneous harmonics, perhaps because he cannot observe any nodes in the movement of the string, which would mean that the concept of nodes is already known, but as too obvious not worth mentioning [Mersenne 1636, Vol. IV, 208–211 see Muzzulini 2006, 129–135]. Although the observable nodes in flageolet tones, which are a kind of isolated overtones, prove a potential complexity of string sounds, they seem to contradict this complexity at the same time, since the shape of the vibrating string does not show any concavities during its movement.

The main difficulty for the physics of vibrations in the 17th and early 18th century is the recognition of the superposition principle, which entails the equivalence of the superposition of sounds and the mathematical sum of functions. Daniel Bernoulli is the first to observe and mathematically describe simultaneous eigenfrequencies in a needle clamped into a wall in 1740 [Cannon et al. 1981, 102], which is an essential step forward in the theory of vibration and its experimental verification.

2. Robert Smith

Robert Smith develops in *Harmonies or the philosophy of musical sounds* [1749] a consonance theory, which applies not only to rational frequency ratios as Leonhard Euler's

gradus suavitatis [Euler 1739] but also to irrational ones. Based on the phenomenon of beats it is strongly opposed to Euler's purely number theoretic approach, and it anticipates Hermann Helmholtz's roughness theory [Helmholtz 1863], as a psychophysical theory on the same fundament with similar implications.
At the same time Jean Lerond d'Alembert, Daniel Bernoulli and Leonhard Euler are working out the general solution for the motion of the ideal string on base of Newton's mechanical laws. In this context Daniel Bernoulli claims that any complex periodic sound can be represented as a superposition of harmonic oscillations and he takes the perception of overtones as evidence for his assertion. He thereby anticipates Ohm's law of acoustics [Bernoulli 1753, 151–153].
The superposition law dicovered by Daniel Bernoulli about 1740 and generalized to non-sinusoidal base functions by Euler [1755, 252] has not been easy to discover as already mentioned. Taylor [1713] for instance believes that the degree of freedom in the motion of the freely vibrating string is only one: According to him, the movement of the plucked string converges within a few periods towards a simple harmonic oscillation – independently of its initial shape. In his argument, he assumes an underlying harmonic oscillation, which performs a kind of attraction to a non-sinusoidal oscillation of the same frequency, and he makes use of difference forces to support his theory. [Cannon et al. 1981, 15–20]

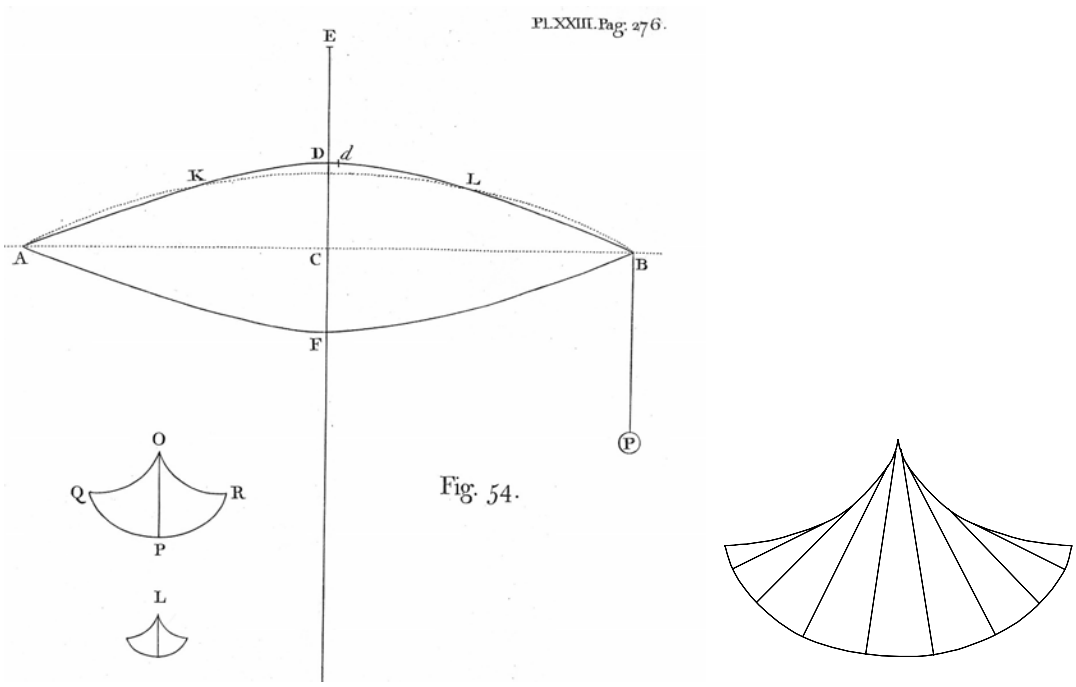


Fig. 2. Asymmetric elongation of a string (AC ≠ BC). Left at the bottom: figures of the cycloidal pendulum. [Smith 1749, 258] Right: Different states in the movement of a cycloidal pendulum. The more it is elongated the shorter is the free length of the swinging cord. This causes its tautochronicity.

In his proof of the periodicity of non-sinusoidal vibrations, Smith argues similarly with difference forces. In contrast to Taylor, however, periodic motion different from sinusoidal vibrations pose no difficulty. He mentions the movement of the cycloidal pendulum as an alternative. But he has no reason to interpret its movement as a combination of sinusoids, since it is not yet known that harmonic vibrations can serve as a mathematical base in the space of periodic functions, as Bernoulli conjectures. And by physical reasons the movement of the cycloidal pendulum seems to be even more "natural" a model for steady tones than the one of the simple pendulum: It is noticed in the 17th century that the time period of an ordinary pendulum is not independent of its initial angle of elongation as Galileo Galilei falsely has claimed. Therefore, the air resistance causes not only the amplitude to decrease in a freely moving pendulum but also a slightly increasing frequency over time[2]. Thus, the damped linear oscillator with its periodic zero-crossings is an imperfect model for the motion of an ordinary pendulum.

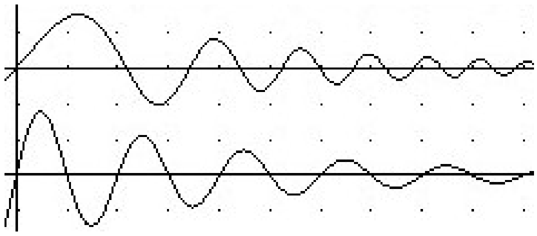


Fig. 3. (a) Damped oscillation with increasing frequency – the zero crossing come closer over time, (b) damped oscillation with equidistant zero-crossings. With respect to zero-crossings an ordinary pendulum behaves more like (a), a damped cycloidal pendulum more like (b).

In order to improve the precision of time measurement Christiaan Huygens develops the cycloidal pendulum in the 1670-is, whose periodicity is independent of elongation, so that it is no longer affected by the air resistance. According to Mahoney [2001] Huygens even "analyzed the string in terms of the cycloid".
Since pitch in a freely vibrating string is essentially independent of duration – only loudness is decreasing –, the analogy of the vibrating string with the cycloidal pendulum appears to be more accurate than its analogy with the ordinary pendulum, as soon as the air resistance is taken into account.

Whereas in the 18th century the vibration theory in its application to the string and the pendulum clearly belongs to the field of continuous mathematics, this does not hold for perception with the same evidence as Smith's treatment of the consonance/dissonance phenomenon shows. Basically, his theory of consonance is a study on superimposed periodic pulse-patterns, which makes of it a discrete theory. In this respect it resembles Euler's onset, which is known to Smith: Euler uses periodic dot sequences to illustrate the varying complexity of simultaneous sounds in his *Tentamen* [1739]. Apart from their frequency ratios, however, Euler's consonance theory does not make any assumptions about the underlying sounds, which justifies the representation of sounds by dot sequences.

The idea of overlaid pulse-patterns goes back to Isaac Beeckman 1614/1615, who has been the first to develop a coincidence theory of consonance [Beeckman 1604–1634/I, 54]. His concept of a tone as a periodic sequence of *ictus* (pulses) is linked to a particle theory of sound propagation, in which the continuous motion of a vibrating string is transformed into a sequence of periodic volleys of flying particles. On the side of the receiver (the inner ear and the brain) these periodic volleys are decoded into pitch sensations according to their time-periodicity. In the middle of the 18th century, however, the wave paradigm of sound propagation seems to be generally accepted [cf. Mairan 1720/1737].

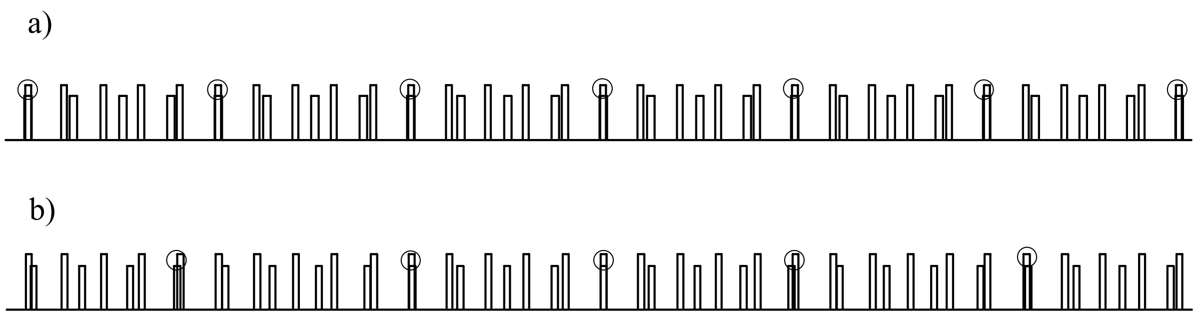


Fig. 4. Two simultaneous periodic pulse functions with different amplitudes and different time-periods. a) In the case of the rational frequency ratio (5 : 4 = 1.25) exactly coinciding pulses marked by circles occur. b) In the case of an irrational frequency ratio (≈ 1.23) only nearly coinciding pulses instead of exactly coinciding pulses occur. They serve Smith to explain the beats for arbitrary frequency ratios. Combined patterns of this kind with irrational frequency ratios never repeat themselves exactly.

In contrast to Euler, Smith does not reject irrational numbers a priori from musical acoustics. Two superimposed periodic functions with arbitrary shapes result in a sum function, which is not periodic at all, if the periodicity ratio is an irrational number – otherwise the sum function is periodic to the least common multiple of the primary periods, see Fig. 4.
This causes a difficulty in determining the beat periodicity of a two-tone-complex, which accounts, according to Smith, for its grade of dissonance. Obviously it cannot be found in the periodicity of the complex signal in case of irrational ratios, because there is none. Nor does the restriction to rational ratios solve the problem: starting with unison a continuous increasing of the periodicity of one of the tones causes a continuously increasing beat frequency, and not a confused jumping to and fro according to the current rational frequency ratios.
In order to solve this problem, Smith uses approximate coincidences of peaks instead of exact coincidences to determine the length of a beating cycle, which is the time between two approximate coincidences of *ictus*. (These approximate coincidences sometimes overlap, if the peaks are not considered as infinitely short.) In Smith's theory the nearly coinciding peaks induce the beat cycles perceived whose frequencies are correlated with the sensation of roughness as in Helmholtz's theory. For a given musical interval it depends on absolute pitch. Smith also mentions a possible influence of timbre on roughness, but he does not treat this aspect mathematically.
Smith's knowledge of the phenomenon of beats goes back to an article by Joseph Sauveur [1700], which he sharply criticizes [Muzzulini 2006, 232–242]. The latter holds beat frequencies up to 6 cps responsible for dissonance sensations and he explicitly points to the dependency of the grade of dissonance from absolute pitch. In this theory the interval of the octave is consonant in the whole range of audible frequencies, because the frequency of the lowest audible sound is bigger than 6 Hz, but smaller intervals as the thirds are less consonant in the lower than in the upper frequency range.
Compared with Smith's theory of pulse patterns, the explanation of the beat phenomenon between pairs of sinusoids of the same amplitude by Helmholtz [1863], which makes use of trigonometric laws, is more elegant. The involved mathematics however becomes complicate, if the involved amplitudes differ.

3. Thomas Young

In *Outline of experiments and inquiries respecting sound and light* [1800] Thomas Young is completely aware of the superposition principle, and he interprets the oscillating movement of the air particles in case of simultaneous sounds explicitly as the mathematical sum of the primary oscillations even for multidimensional vibrations. He also criticizes Smith in this respect. This understanding leads him to the probably first modern visualization of the beats between pairs of tones. Thereby he discusses not only sinusoidal but also triangular oscillations as basic movement types. The superposition of two of these base functions yields the same behavior of periodically increasing and decreasing amplitudes as sine functions, see Fig. 5.

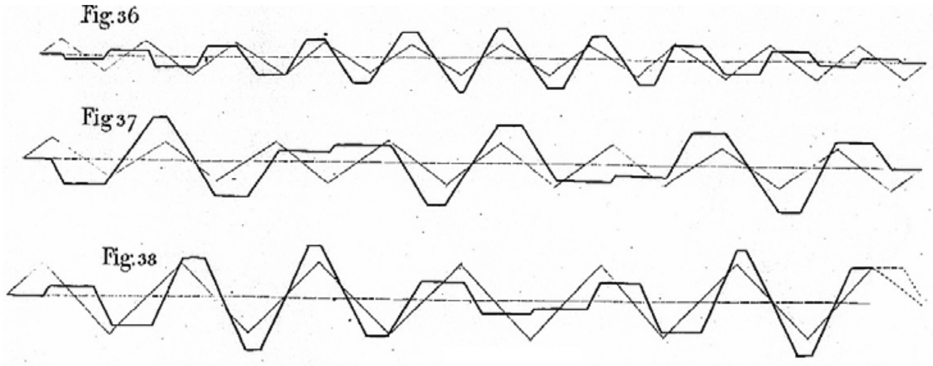


Fig. 5. Two overlaid triangular oscillations result in a modulating trapezoid oscillation. Only one of the primary triangular oscillations is shown. [Young 1800, Plate 5, 150]

Young exemplifies the transmission of sound as the result of the interaction of oscillating air particles having a constant speed. In this model, reflections with no loss of speed occur when two particles collide[3].

> [...] let us suppose, what probably never precisely happens, that the particles of air, in transmitting the pulses, proceed and return with uniform motions; and, in order to represent their position to the eye, let the uniform progress of time be represented by the increase of the absciss, and the distance of the particle from its original position, by the ordinate [Young 1800, 131]

Indeed, according to Newton's laws, no loss of velocity could occur only if the particles were periodically reflected at pairs of parallel plates at rest or if the particles moved along the same line in opposite direction, but not if a moving particle collided with a particle at rest or if they had different directions. Nevertheless, Young's representation is much more than just a drawing simplification for sine functions. And it is not a priori clear, whether the propagated air particles follow Newton's laws valid for solid bodies.

Most probably, Young has been inspired by a remark of Euler [1755, 252; see Muzzulini 2006, 211–212] to the following considerations, which bring into question the special status of the sinusoids from a mathematical point of view.

> It is remarkable, that the law by which the motion of the particles is governed, is capable of some singular alterations by a combination of vibrations. By adding to a given sound other similar sounds, related to it in frequency as the series of odd numbers, and in strength inversely in the same ratios, the right lines indicating an uniform motion may be converted very nearly into figures of sines, and the figures of sines into right lines, as in Plate V. Figs. 39, 40. [Young 1800, 133]

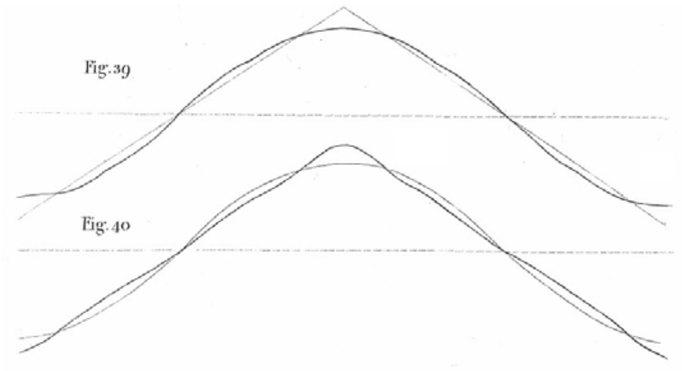


Fig. 6 Young's triangular and sinusoidal synthesis [Young 1800, Plate 5, 150] with his own legends:
Fig. 39. A vibration of a similar nature [uniform speed, triangular oscillation dm], combined with subordinated vibrations of the same kind in the ratios of 3, 5, 7.
Fig. 40. A vibration represented by a curve of which the ordinates are the sines of circular arcs increasing uniformly, corresponding with the motion of a cycloidal pendulum, combined with similar subordinate vibrations in the ratios of 3, 5 and 7. [Young 1800, 148–149]

The legend to Young's Fig. 40 seems to contradict the description in the main text. The phrase "sines of circular arcs increasing uniformly" defines the mathematical sine function unambiguously: The length of a "circular arc" is equal to the angle in the radian measure. If the angle is "increasing uniformly" with time *t*, it is of the form νt, and the corresponding movement is of the form $\sin(\nu t)$, i.e. an ordinary sinusoidal and not a cycloidal vibration. Furthermore, the base curve represented in Fig. 40 cannot be a cycloid, which can be seen at its zero-crossings, where a true cycloid would have a vertical tangent. Thus, the phrase "corresponding with the motion of a cycloidal pendulum" is misleading and if it is cancelled everything is fine.

Young gives nothing but the Fourier decomposition of a triangular vibration *triang*(*x*) which is correct, if "strength" is taken as square of amplitudes:

$$triang(x) \approx \sin(x) - \tfrac{1}{9}\cdot\sin(3x) + \tfrac{1}{25}\cdot\sin(5x) - \tfrac{1}{49}\cdot\sin(7x) \qquad \text{(A)}$$

This function is represented in Fig. 7a, which is in good accordance with Young's Fig. 40.

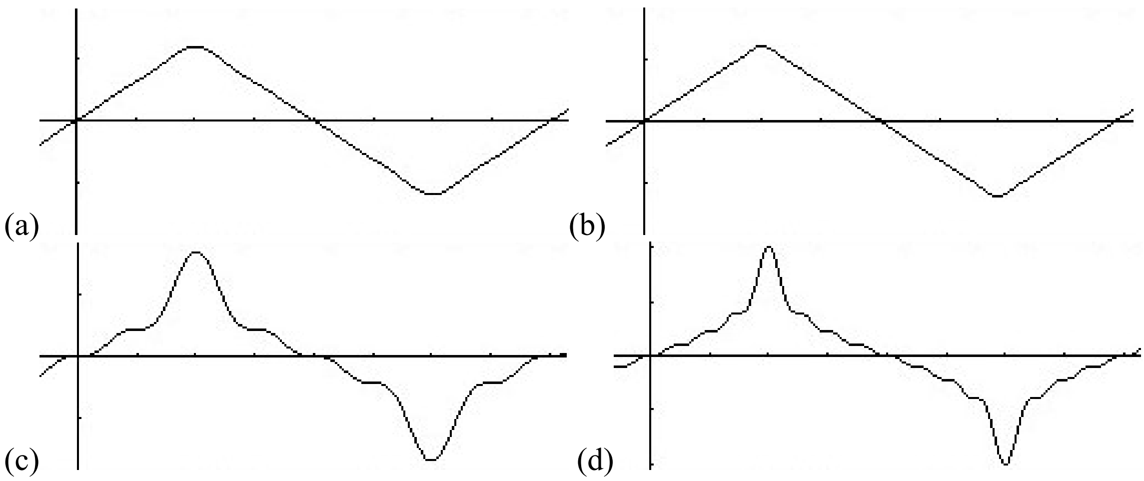


Fig. 7. Fourier synthesis according to Young.

(a) $triang(x) \approx \sin(x) - \tfrac{1}{9}\cdot\sin(3x) + \tfrac{1}{25}\cdot\sin(5x) - \tfrac{1}{49}\cdot\sin(7x)$ (4 components)

(b) $triang(x) \approx \sum_{n=0}^{7} \frac{(-1)^n}{(2n+1)^2}\cdot\sin((2n+1)\cdot x)$ (8 components)

(c) $\sin(x)-\frac{1}{3}\cdot\sin(3x)+\frac{1}{5}\cdot\sin(5x)-\frac{1}{7}\cdot\sin(7x)$

(d) $\sum_{n=0}^{7}\frac{(-1)^n}{2n+1}\cdot\sin((2n+1)\cdot x)$

With the amplitudes inversely proportional to the square of frequencies as in (a) and (b) the desired approximation of the triangular oscillation is achieved, whereas the amplitudes inversely proportional to the frequencies as in (c) and (d) generate qualitatively different wave shapes: the peaks are not even limited, if the number of components tends to infinity.

—

Young's Fig. 39 on the other hand gives a triangular decomposition of the sinusoidal vibration, which is

$$\sin(x)\approx triang(x)+\tfrac{1}{9}\cdot triang(3x)-\tfrac{1}{25}\cdot triang(5x)+\tfrac{1}{49}\cdot triang(7x) \qquad \text{(B)}$$

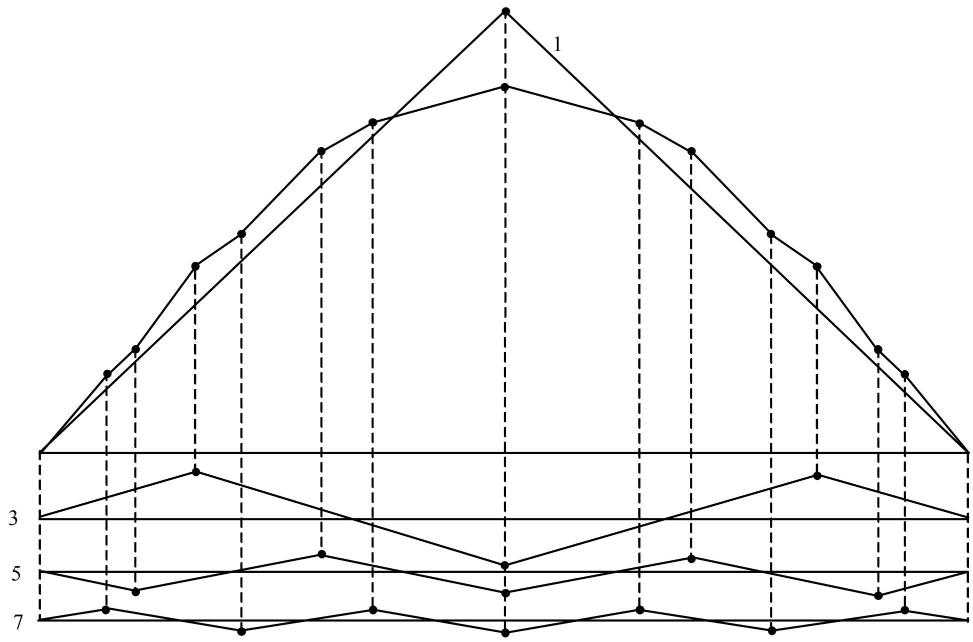


Fig. 8. Construction of Young's triangular synthesis of the sine function, compare with Young's Fig. 39. The triangular fundamental (1) plus the components to the third, fifth and seventh triangular harmonics (3,5 and 7) sum up to a piecewise linear approximation of a sinusoid. Whereas the formula A is completely regular also for higher order terms, formula B is not! The mathematical analysis shows that the coefficients of the 9th triangular component is 0 and not –1/81, and the coefficients of the 15th is -1/1125 and not 1/225, as could be expected.

It is unclear how, if not heuristically, Young has detected the representations A and B. The calculation of the two series and the proof of their convergence are by no means trivial. The method to determine the coefficients in A is developed by Fourier in 1811, but not accepted for publication until 1822. The author's deduction of B makes use of the Fourier series A and of matrix algebra. The representation of an arbitrary periodic continuous function $f(t)$ by a finite sum of triangular vibrations is performed by piecewise linear functions on a non-uniform node set, which is an approximation and not an interpolations of the given function. By increasing the number of components these approximations converge point-wise towards $f(t)$, although they do not share essential properties such as differentiability and convexity with the approximated function $f(t)$, see Fig 8.

A similar analysis and synthesis can be done not only with sinusoidal and triangular base functions, but also with almost arbitrary shaped periodic functions, as Euler rightfully claims [Euler 1755, 252]. Hence, the use of sinusoids as a base for periodic sounds is nothing but a mathematically comfortable choice. If no natural unique and elementary waveform exists, which seems to follow from the above considerations, the distinction between atomic and complex sounds loses its objectivity.
From this point of view, Young's explanation of "Tartini's sounds" as rapid beats turned into sounds appears to be cogent, since the only property of a sound that could determine its pitch independently of the underlying waveform is its periodicity:

> The greater the difference in the pitch of two sounds, the more rapid the beats, till at last, like the distinct puffs of air in the experiments already related, they communicate the idea of a continued sound; and this is the fundamental harmonic described by TARTINI. For instance, in the Plate V. Fig. 34–37, the vibration of sounds related as 1 : 2, 4 : 5, 9 : 10, and 5 : 8, are represented; where the beat, if the sounds be not taken too grave, constitute a distinct sound, which corresponds with the time elapsing between two successive coincidences, or near approaches to coincidence: for, that such a tempered interval still produces a harmonic, appears from Plate V. Fig. 38. [Young 1800, 132] [see Fig. 5]

He interprets "Tartini's sounds" as the perception of the periodicity of the compound waveform of two periodic vibrations with a rational frequency ratio. The wording in the second part of the argument "successive coincidences, or near approaches to coincidence" about irrational frequency ratios reminds of Robert Smith. According to Young, the Tartini sounds are a kind of residual tones [Schouten 1940] and not combination tones in the sense of Hällström [1832] and Helmholtz [1863], which originate in a non-linear transmission. Pitch perception according to Young is not restricted to a specific vibration pattern; it is nothing but the perception of periodicity. This point of view sharply contrasts with the approach of Daniel Bernoulli and Ohm, where an elementary tone is the sensation caused by a harmonic, sinusoidal oscillation. Young's onset is more compatible with the viewpoints of Euler [1755], Willis [1830], Seebeck [1844] and Koenig [1881b], which let sound quality depend on the time pattern of vibrations and which equate pitch with periodicity.

4. William Thomson (Lord Kelvin)

The question of periodicity pitch, i.e. pitch perception with no corresponding primary frequencies in form of sinusoidal components, seems to be much easier to be understood with no *Theorem of Fourier* at hand that would allow to decide if a tone can be considered as a "real part" ["reeller Bestandtheil": Ohm 1843] of a periodic sound or not. Surprisingly, the controversy about the nature of tones is not decided by Fourier's "beautiful analysis" [Thomson 1877/78], although its completing proof by Dirichlet [1829] closes an important gap in mathematics. The discussion just restarts, even with more impetus, it culminates in the famous Ohm/Seebeck-dispute, and it still lasts after Helmholtz [1863] establishes his resonance theory of the basilar membrane.
Experimenting with dyads of tuning fork sounds (amplified by cylindrical resonators) Koenig asserts the existence of higher order beats, which are beats between non unison sinusoids, and he observes a new kind of tone sensation, which he calls "Stosston" (pushing sound), which is not present in the two sound sources [Koenig 1876]. These results are hardly compatible with Helmholtz's proclaimed insensitivity for phase relationships (outside a critical bandwidth). Therefore, Helmholtz explains them as beats involving difference tones or subjective harmonics [cf. Koenig 1881, 336]. Koenig, however, tries to show that these effects are not caused by non-linear distortion, that they can be explained neither as combination tones nor by the interaction of subjective overtones with "real" harmonics in the inner ear. Subjective overtones are non-linear distortion products of a loud sinusoid. They "exist" in the cochlea as ordinary tones, but without a corresponding frequency component in the sound source.
To support Koenig's findings, William Thomson experiments with a set of Koenig's tuning forks. He considers them to be free of overtones, and he develops a theory of higher order beats, which is essentially the same as the one of Smith outlined above. For modeling the superposition of two sinusoidal vibrations, so called binary harmonies, he also recurs to a discrete onset. Differently to Smith he models one cycle of a pure tone as a pair of peaks in opposite directions forming rotation symmetric discrete signals, which are the discrete version of both the sinusoidal and the triangular vibration. The idea of nearly coinciding peaks is thereby worked out as an alternation of reinforcing and annihilating coincidences.

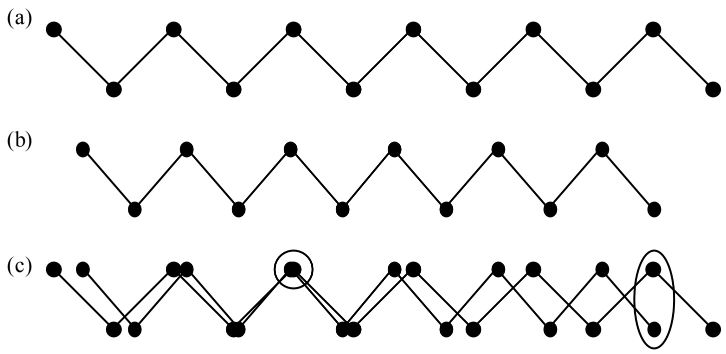


Fig. 9. (a) and (b): tones as pulse-patterns with alternating sign and different periodicity according to Thomson. (c): the patterns (a) and (b) overlaid. The marked places of coincidence and opposition correspond with reinforcement and annihilation and are used to explain the beating phenomenon. Thomson only studies the dots and not their connections cf. Young in Fig. 5.

Since they have an infinite set of sine components in their Fourier decomposition, periodic pulse functions cannot prove higher order beats between sinusoids. Thomson's discrete onset only develops a method that allows detecting peaks and zeros in the envelope of complex sounds.

In the 20th century the concept of critical bandwidth, which goes back to Helmholtz [1863] leads Schouten [1940] to an explanation of Koenig's Stosston as a residue: Within a critical band, where two or more sinusoidal components are not transmitted independently to the nervous system because of their interaction on the basilar membrane, the time pattern of a complex tone can cause a tone sensation, which sensibly depends on the phase relationships between its components. On the basilar membrane this tone sensation is triggered in the zone of the interacting components, and not at the place of the missing fundamental corresponding to the pitch perceived, see Fig. 10.

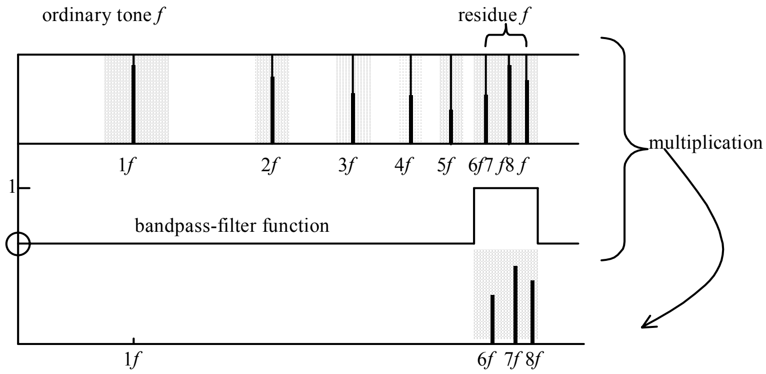


Fig. 10. Ideal band pass filter extracting the harmonics 6*f*, 7*f*, 8*f* from a complex periodic sound. The extracted components give a signal with the same time-period $T = 1/f$ as they original sound. If two or more harmonics of a complex sound lie within the same critical band, they are not independently transmitted to the brain. In this case they form a "percept of unresolved harmonics" [Schouten et al. 1962, 1418–1419], which can cause a pitch perception corresponding with its time-period, although it contains no frequency component 1*f*. The thereby perceived tone is called a *residual tone* or *residue*, and it has a sharp timbre. Residual tones do not "exist" on the basilar membrane, since they do not interfere with ordinary tones.

The limited frequency resolution of the basilar membrane cannot explain beats between pure tones that are separated by more than one critical band. The resonance places of two sinusoids forming a musical fifth (frequency ratio 3 : 2) with moderate intensity lie more than a critical band apart. In this case there is no residue to the missing fundamental 1, and if the interval is slightly mistuned (as in the 12-tempered tuning), there are no beats possible according to the theory of the residue.
Therefore, theories of consonance/dissonance taking roughness resulting from beating sinusoidal components as the primary cause of dissonance [cf. Plomp et al. 1975] lead to musically implausible results, if they are applied to sounds having the "natural waveform of sinusoids": because of a lack of beats the fifth, the fourth and the diminished fifth – the devil in music – are equally consonant!

The examined sources show that the question of the natural time pattern of a tone can be decided only in an ambiguous way in favor of the harmonic oscillation.

Footnotes

1) The concept of natural frequencies is developed by Galileo Galilei [1638, 141–143], who also explains mutual resonance by periodic vibrations transmitted as time-periodic pulses of the air in the same way as Beeckman (1614). Furthermore he gives some examples of Chladni-figures [Galilei 1638, 144–145]
2) The approximation $\alpha = \sin \alpha$ is valid only for small angles α [see Neukom 2003, 190–193, with animation].
3) For a short introduction to the modeling of particles and their movements in the kinetic theory of gases, founded by Daniel Bernoulli (1738), see Gowers 2002, 8–11.

References

[Barkowsky 1996]	Johannes Barkowsky: Das Fourier-Theorem in musikalischer Akustik und Tonpsychologie. Schriften zur Musikpsychologie und Musikästhetik, Bd. 8., Ed. Helga de la Motte-Haber, Peter Lang, Frankfurt am Main, 1996
[Beeckman 1604–1634]	Loci Communes, Journal tenu par Isaac Beeckman de 1604 à 1634, Ed. C. de Waard, 4 Bände, Martinus Nijhoff, La Haye 1939
[Bernoulli 1753]	Daniel Bernoulli, Réflexions et éclaircissemens sur les nouvelles vibrations de cordes, Histoire de l'Académie des Sciences et Belles-Lettres 9 (1753), 147–152
[Cannon et al. 1981]	John T. Cannon, Sigalia Dostrovsky, The evolution of dynamics: vibration theory from 1687 to 1742, Studies in the history of mathematics and physical sciences, Springer, New York 1981
[Dirichlet 1829]	Peter Gustave Leujeune Dirichlet, Sur la convergence des series trigonometriques qui servent a representer une fonction arbitraire entre des limites donnees, 1829
[Euler 1739]	Leonhard Euler, Tentamen novae theoriae musicae, 1739, Opera Omnia, Serie 3, Vol. 1, 197–427
[Euler 1755]	Leonhard Euler, Remarques sur les Mémoires précédens de M. Bernoulli,1755, Opera Omnia, Serie 2, Vol. 10, 1947, 233–254
[Fourier 1811]	J. B. Fourier, Mém. acad. France 4, 185, Paris 1924 (handed in 29.9.1811)
[Fourier 1822]	J. B. Fourier, Théorie analytique de la chaleur, Paris 1822
[Galilei 1638]	Galileo Galilei, Discorsi e Dimostrazioni Matematiche, intorno à due nuove scienze 1638. Dialogues Concerning Two New Sciences, Ed. Stephen Hawking, Running Press, Philadelphia 2002
[Hällström 1832]	Gustav Gabriel Hällström, Von den Combinationstönen, Poggendorff's Annalen, 24, 1832, 438–467
[Helmholtz 1863]	Hermann von Helmholtz, Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik, Vieweg, Braunschweig 1863
[Koenig 1876]	Rudolph Koenig, Über den Zusammenklang zweier Töne, Annalen der Physik 157, 1876, 177–237
[Koenig 1881]	Rudolph Koenig, Über den Ursprung der Stöße und Stoßtöne bei harmonischen Intervallen, Annalen der Physik 12, 1881, 335–349
[Koenig 1881b]	Rudolph Koenig, Bemerkungen über die Klangfarbe, Annalen der Physik und Chemie, 14, 1881, 369–393
[Mahoney 2001]	Michael S. Mahoney (Princeton University), Sketching science in the seventeenth century. < http://www.princeton.edu/~hos/mike/articles/whysketch/whysketch.html>
[Mairan 1720]	Jean-Jacques Dortus de Mairan, Diverses observations de physique generale. IV. Histoire de l'Académie Royale des Sciences, 1720, Avec les Memoires de Mathematique & de Physique, pour la même Année. Paris 1722;11–12
[Mairan 1737]	Jean-Jacques Dortus de Mairan, Discours sur la Propagation du Son dans les differents Tons qui le modifient. Memoires de Mathematique & de Physique 1737, 1–20] Eclaircissments sur le Discour Precedent. Memoires de Mathematique & de Physique 1737, 20 – 60
[Mersenne 1636]	Marin Mersenne, Harmonie Universelle, Paris 1636, Reprint: Centre National de la Recherche Scientifique, Paris 1975
[Muzzulini 2006]	Daniel Muzzulini, Genealogie der Klangfarbe, Varia Musicologica Vol. 5, Peter Lang, Bern, 2006
[Neukom 2003]	Martin Neukom, Signale, Systeme und Klangysnthese, Grundlagen der Computermusik, Zürcher Musikstudien, Forschung und Entwicklung an der HMT Zürich, Band 2, Peter

Lang, Bern 2003

[Ohm 1839] Georg Simon Ohm, Bemerkungen über Combinationstöne und Stöße, Poggendorf's Annalen der Physik und Chemie, Band XXXXVII, 1839, 463–466

[Ohm 1843] Georg Simon Ohm, Ueber die Definition des Tones, nebst daran geknüpfter Theorie der Sirene und ähnlicher tonbildender Vorrichtungen, Poggendorf's Annalen der Physik und Chemie, Band LIX, 1843, 497–565

[Ohm 1844] Georg Simon Ohm, Noch ein Paar Worte über die Definition des Tones, Poggendorf's Annalen der Physik und Chemie, Band LXII, 1844, 1–18

[Plomp et al. 1975] Reinier Plomp, Wihem J.M. Levelt, Tonal consonance and critical bandwidth, Journal of the acoustical society of America, Vol. 68, 1965, 548–560

[Rameau 1737] Génération harmonique, 1737, Jean-Philippe Rameau, Complete Theoretical Writings, hg. Erwin Jacobi, Vol. 3, American Institute of Musicology, 1967

[Sauveur 1700] Joseph Sauveur, Sur la détermination d'un son fixe. Histoire de l'Academie Royale des Sciences. Année 1700, Seconde Edition, revue, corrigé & augmentée, Chez Pierre Mortier, Amsterdam 1734, 182–195

[Schouten 1940] J. F. Schouten, The Residue a New Component in Subjective Sound Analysis, Proc. Koinkl. Ned. Akad. Wetenschap. 43, 1940, 356–365

[Schouten et al. 1962] J. F. Schouten, R. J. Ritsma, B. Lopes Cardozo, Pitch of the Residue, JASA 34, 1962, 1418–1424

[Seebeck 1844a] August Seebeck, Ueber die Definition des Tones, Poggendorf's Annalen der Physik und Chemie, Band LXIII 1844, 353–368

[Seebeck 1844b] August Seebeck, Ueber die Erzeugung von Tönen durch getrennte Eindrücke, mit Beziehung auf die Definition des Tones, Poggendorf's Annalen der Physik und Chemie, Band LXIII 1844, 368–380

[Smith 1749] Harmonics, or the philosophy of musical sounds, Cambridge 1749

[Tartini 1754] Giuseppe Tartini, Trattato di musica secondo la vera scienza dell' armonia, Padova 1754, Reprint, Novecento, Palermo 1996

[Thomson 1877/78] William Thomson, On beats of imperfect harmonies, Proceedings of the Royal Society of Edinburgh, Session 1877–78, Vol. IX, 602–612

[Taylor 1713] Brook Taylor, De motu nervi tensi, Phil. Trans. Roy. Soc. London 27, 1713, 26–32

[Wallis 1677] Dr. Wallis's Letter to the Publisher, concerning a new Musical Discovery; written from Oxford, March 14. 1676/7, Philosophical Transactions, The Royal Society of London, Vol. 12, 1677, Johnson Reprint Corporation, Kraus Reprint Corporation, New York 1963, 840–842

[Willis 1830] Robert Willis, On the vowel sounds, and on the reed-organ pipes, Transactions of the Cambridge Philosophical Society, Vol. 3, 1830, 229–268

[Young 1800] Thomas Young, Outline of experiments and inquiries respecting sound and light, Phil. Trans. Roy. Soc. London 1, 1800, 106–150